

# TERMINAL-PAIRABILITY IN COMPLETE GRAPHS

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**ABSTRACT.** We investigate terminal-pairability properties of complete graphs and improve the known bounds in two open problems. We prove that the complete graph  $K_n$  on  $n$  vertices is terminal-pairable if the maximum degree  $\Delta$  of the corresponding demand multigraph  $D$  is at most  $2\lfloor \frac{n}{6} \rfloor - 4$ . We also verify the terminal-pairability property when the number of edges in  $D$  does not exceed  $2n - 5$  and  $\Delta \leq n - 1$  holds.

*Dedicated to the memory of our friend, professor Ralph Faudree.*

## 1. INTRODUCTION

We discuss a graph theoretic concept of *terminal-pairability* emerging from a practical networking problem introduced by Csaba, Faudree, Gyárfás, Lehel, and Shelp [1] and further studied by Faudree, Gyárfás, and Lehel [2, 3, 4] and by Kubicka, Kubicki and Lehel [5]. We revisit two open problems presented in [1] and [5]. Let  $G$  be a graph with vertex set  $V(G) = T(G) \cup I(G)$  where the set  $T(G)$  consists of  $t$  ( $t$  even) vertices of degree 1. We call  $G$  a *terminal-pairable* network if for any pairing of the vertices of  $T(G)$  there exist edge-disjoint paths in  $G$  between the paired vertices.  $T(G)$  is referred to as the set of *terminal nodes* or *terminals* and  $I(G)$  is called the set of interior nodes of the network. Given a particular pairing of the terminals, the pairs of terminals in the pairing are simply called *pairs*. For an inner vertex  $v$  we denote the number of terminal and interior vertices incident to  $v$  by  $d_{T(G)}(v)$  and  $d_{I(G)}(v)$ , respectively.

In a terminal-pairable network pairs of vertices of a graph are to be connected with edge-disjoint paths, thus the notion is clearly related to

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multicommodity flow problems. The concept is also related to weakly-linked (in our case weakly- $t/2$ -linked) graphs: a graph  $G$  is *weakly  $k$ -linked* if, for every pair of  $k$ -element sets,  $X = \{x_1, \dots, x_k\}$  and  $Y = \{y_1, \dots, y_k\}$ , there exist edge-disjoint paths  $P_1, \dots, P_k$ , such that each  $P_i$  is an  $x_i y_i$ -path. Observe that joining terminal vertices (leaves) to the vertices of a weakly- $k$ -linked graph  $G$  results in a terminal-pairable graph as long as every vertex of  $G$  receives at most  $k$  terminals. On the other hand, note that terminal-pairable graphs are not necessarily highly-weakly-linked. The stars (complete bipartite graph  $K_{1,n}$  where one class is formed by a singleton) give a very illustrative example of terminal-pairable graphs with many terminal vertices that are not even weakly-2-linked.

Given a terminal-pairable network  $G$  with a particular pairing  $\mathcal{P}$  of the terminals the demand multigraph  $D = (V(D), E(D))$  is defined as follows: we set  $V(D) = I(G)$  and join two vertices  $u, v \in V(D)$  by as many copies of the edge  $uv$  as there are pairs of terminals in  $\mathcal{P}$  s.t. one vertex of the pair is joined to  $u$  and the other is joined to  $v$  in  $G$ . Obviously,  $|E(D)| = \frac{|T(G)|}{2}$  and  $d_D(v) = d_{T(G)}(v)$  for every  $v \in V(D)$ , thus in fact  $\Delta(D) = \max\{d_{T(G)}(v) \mid v \in I(G)\}$ . For convenience, demand multigraphs are referred to simply as demand graphs from now on.

Observe that a terminal pairing problem is fully described by the underlying network  $G$  and the demand graph  $D$ . We call the process of substituting the demand edges by disjoint paths in  $G$  the *resolution of the demand graph*.

Given a simple graph  $G$ , one central question in the topic of terminal-pairability is the maximum value of  $t$  for which an arbitrary extension of  $G$  by  $t$  terminal nodes results in a terminal-pairable graphs. As at a given vertex  $v \in I(G)$  at most  $d_{I(G)}(v)$  edge-disjoint paths can start, the minimum degree  $\delta_{I(G)}$  of the graph induced by the interior vertices provides an obvious upper bound on  $t$ . However, with a balanced placement of the terminals with restriction on  $\Delta(D)$  of the corresponding demand graph (resembling the structure of weakly-linked graphs), the  $\delta_{I(G)}$  bound on the extremal value of  $t$  can be greatly improved.

Csaba, Faudree, Gyárfás, Lehel, and Shelp [1] studied above extremal value for the complete graph  $K_n$  and investigated the following question:

**Problem 1 ([1]).** *Let  $K_n^q$  denote the graph obtained from the complete graph  $K_n$  ( $n$  even) by adding  $q$  terminal vertices to every initial vertex. What is the highest value of  $q$  (in terms of  $n$ ) for which  $K_n^q$  is terminal-pairable?*

One can easily verify that the parameter  $q$  cannot exceed  $\frac{n}{2}$ . Indeed, take the demand graph  $D$  obtained by replacing every edge in a one-factor on  $n$  vertices by  $q$  parallel edges. In order to create edge-disjoint paths most paths need to use at least two edges in  $K_n$ , thus a rather short calculation

implies the indicated upper bound. The so far best result on the lower bound is due to Csaba, Faudree, Gyárfás, Lehel, and Shelp:

**Theorem 2** (Csaba, Faudree, Gyárfás, Lehel, Shelp [1]). *If  $q \leq \frac{n}{4+2\sqrt{3}}$ , then  $K_n^q$  is terminal-pairable.*

We improve their result by proving the following theorem:

**Theorem 3.** *If  $q \leq 2\lfloor \frac{n}{6} \rfloor - 4$ , then  $K_n^q$  is terminal-pairable.*

Kubicka, Kubicki and Lehel [5] investigated terminal-pairability properties of the Cartesian product of complete graphs. In their paper the following “Clique-Lemma” was proved and frequently used:

**Lemma 4** (Kubicka, Kubicki, Lehel [5]). *Let  $G$  be a complete graph on  $n$  vertices, where  $n \geq 5$ . If every vertex of  $G$  has at most  $n - 1$  adjacent terminals and the total number of terminals is  $2n$ , then for every pairing of terminals, there are edge disjoint paths for all pairs.*

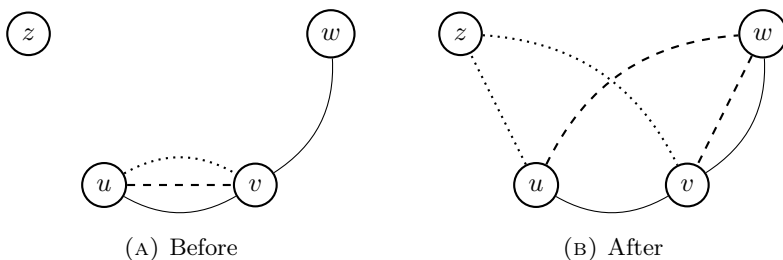
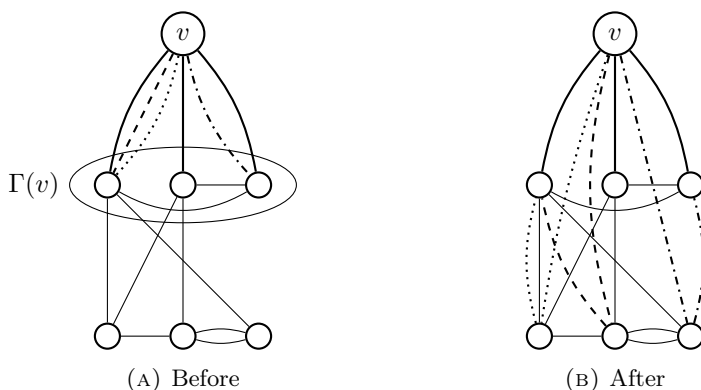
In the same paper the following related problem was raised about the possible strengthening of Lemma 4:

**Problem 5** ([5]). *Find the largest value of  $\alpha$  such that  $K_n$  with  $\alpha \cdot n$  terminals (at most  $n - 1$  at each vertex) has the above property for all  $n$  larger than some constant  $n_0$ .*

Obviously,  $2 \leq \alpha$  due to Lemma 4. It is also easy to see that  $\alpha < 4$ . Let  $D$  be a demand graph on  $n \geq 4$  vertices, in which two pairs of vertices,  $U, V$  and  $X, Y$  are both joined by  $(n - 2)$  parallel edges ( $2n - 4$  edges or equivalently  $4n - 8$  terminals in total;  $d_D(W) = 0$  for  $W \notin \{X, Y, U, V\}$ ). Observe that to resolve the demand graph any disjoint path system must contain a path from  $X$  to  $Y$  passing through  $U$  or  $V$ . However, there are also  $n - 2$  disjoint paths connecting  $U$  and  $V$ , meaning that  $U$  or  $V$  is incident to at least  $2 + (n - 2) = n$  disjoint edges, which is clearly a contradiction. This implies that the number of terminals in  $G$  cannot exceed  $4n - 10$ . We show that this bound is sharp by proving the following theorem:

**Theorem 6.** *Let  $D$  be a demand graph with at most  $2n - 5$  edges such that no vertex is incident to more than  $n - 1$  edges. Then  $D$  can be resolved.*

Before the proofs we fix further notation and terminology. For convenience, we call a pair of edges joining the same two vertices a  $C_2$ . For  $k > 2$ ,  $C_k$  denotes the cycle on  $k$  vertices. For a subset  $S \subset V(G)$  of vertices let  $d(S, V(G) - S)$  denote the number of edges with exactly one endpoint in  $S$ . Let  $[S]$  denote the subgraph induced by the subset of vertices  $S$ . We call a pair of vertices joined by  $k$  parallel edges a  $k$ -bundle. For a vertex  $v$  we denote the set of neighbors by  $\Gamma(v)$  and use  $\gamma(v) = |\Gamma(v)|$ . We define the *multiplicity*  $m(v)$  of a vertex  $v$  as follows:  $m(v) = d(v) - \gamma(v)$ . Observe


 FIGURE 1. Lifting 2 edges of  $uv$  to  $z$  and  $w$ 

 FIGURE 2. Resolving the multiplicities at  $v$ 

that  $m(v)$  is the minimal number of direct edges that need to be replaced by longer paths in the graph to guarantee an edge-disjoint path-system for the terminals of  $v$ .

We define an operation that we will subsequently use in our proofs: given an edge  $uv \in E$  we say that we *lift*  $uv$  to a vertex  $w$  when substituting the edge  $uv$  by a path of consecutive edges  $uw$  and  $wv$ . Note that this operation increases the degree of  $w$  by 2, but does not affect the degree of any other vertex (including  $u$  and  $v$ ). Also, as a by-product of the operation, if  $w$  is already joined by an edge to  $u$  or  $v$ , the multiplicity of the appropriate pair increases by one (see Figure 1).

Finally, note that if a graph  $G$  has  $n$  vertices and  $d(v) \leq n - 1$ , all multiplicities of  $v$  can be easily resolved by subsequent liftings. Indeed,  $v$  has  $n - 1 - \gamma(v)$  non-neighbors and  $m(v) = d(v) - \gamma(v) \leq n - 1 - \gamma(v)$  multiplicities, thus we can assign every edge of  $v$  causing a multiplicity to a non-neighbor to which that particular edge can be lifted. We call this *resolution of the multiplicities of  $v$*  (see Figure 2).

## 2. PROOF OF THEOREM 3

We show that if  $D = (V, E)$  is a demand multigraph on  $n$  vertices and  $\Delta(G) \leq 2\lfloor \frac{n}{6} \rfloor - 4$ , then  $D$  can be transformed into a simple graph by replacing parallel edges by paths of  $D$ . We prove the statement by induction on  $n$ . Observe first that the statement is obvious for  $n < 18$ . For  $18 \leq n < 24$ , note that the demand graph  $D$  is the disjoint union of 2-bundles, circles, paths, and isolated vertices. It is easy to see that multiplicities in these demand graphs can be resolved; we leave the verification of the statement to the reader.

From now on assume  $n \geq 24$ . We may assume without loss of generality that  $D$  is an  $(2\lfloor \frac{n}{6} \rfloor - 4)$ -regular multigraph; if necessary, additional parallel edges may be added to  $D$ . Should a single vertex  $v$  fail to meet the degree requirement, we bump up its degree by further lifting operations as follows: as the deficit  $(2\lfloor \frac{n}{6} \rfloor - 4) - d(v)$  must be even, we can lift an arbitrary edge  $e \in E([V(D) - v])$  to  $v$ . We remind the reader that lifting  $e$  to  $v$  increases  $d(v)$  by two while it does not affect the degree of the rest of the vertices.

We will use the well known 2-Factor-Theorem of Petersen [6]. Be aware that a 2-factor of a multigraph may contain several  $C_2$ 's (however, this is the only way parallel edges may appear in it).

**Theorem 7** ([6]). *Let  $G$  be a  $2k$ -regular multigraph. Then  $E(G)$  can be decomposed into the union of  $k$  edge-disjoint 2-factors.*

Some operations, which are performed later in the proof, are featured in the following definition, claim, and lemma.

**Definition 8** (Lifting coloring). Let  $F$  be a multigraph, and  $c : E(F) \cup V(F) \rightarrow \{1, 2, 3\}$  be a coloring of the edges and vertices of  $F$ . We call  $c$  a *lifting coloring* of  $F$  if and only if

- (1) for any edge  $e = uv \in E(F)$ ,  $c(u) \neq c(e)$  and  $c(v) \neq c(e)$ , and
- (2) for any two edges  $e_1, e_2 \in E(F)$  incident to a common vertex we have  $c(e_1) \neq c(e_2)$ .

Moreover, if the number of vertices in different color classes differ by either 0, 1, or 2, then we call  $c$  a *balanced lifting coloring* of  $F$ .

**Claim 9.** *Let  $F$  be a multigraph such that  $\forall v \in V(F)$  we have  $d_F(v) \leq 2$ . If  $w_1, w_2, w_3 \in V(F)$  are three pairwise non-adjacent different vertices, then  $F$  has a balanced lifting coloring where  $w_i$  gets color  $i$ .*

*Proof.* The proof is easy but its complete presentation requires a rather lengthy (but straightforward) casework. We leave the verification of the statement to the reader. Figure 3 shows an example output of this lemma.  $\square$

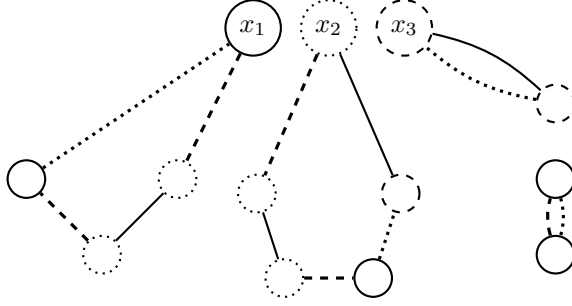


FIGURE 3. A balanced lifting coloring, where  $x_1, x_2, x_3$  get pairwise different colors.

**Lemma 10.** *Let  $D$  be a demand graph on  $n$  vertices, such that  $\Delta(D) \leq \lfloor \frac{n}{3} \rfloor - 4$ . Furthermore, let  $X = \{x_1, x_2, x_3\}$  be a subset of  $V(D)$  of cardinality 3, such that  $|E(D[X])| = 0$ . Let  $B$  be an at most 3 element subset of  $V(D) \setminus X$ . Let  $F$  be an  $\leq 2$ -factor of  $D$ . Then there exists a demand graph  $H$  which satisfies*

- $V(H) = V(D) \setminus X$ ,
- $E(H) \supset E(D[V(H)]) \setminus F$ ,
- $\{e \in E(H) : e \text{ is incident to at least one of } B\} \subset E(D)$ , and
- for any  $v \in V(H)$  we have  $d_H(v) \leq d_D(v) - d_F(v) + 1$  if  $v \notin B$ .

Moreover, if  $H$  has a resolution, then so does  $D$ .

*Proof.* We will perform a series of liftings in  $D$  in two phases, obtaining  $D'$  and  $D''$ . At the end of the second phase, we will achieve that  $X$  has no parallel edges in  $D''$ . Therefore setting  $H = D'' - X$  will satisfy the second claim of the lemma.

First, we determine the series of liftings to be executed in the first phase. Notice that Claim 9 implies the existence of a balanced lifting coloring  $c$  of  $F$  such that  $c(x_i) \equiv i + 1 \pmod{3}$ . Lift each edge  $f \in F$  to  $x_{c(f)}$ , except if  $f$  is incident to  $x_{c(f)}$ . Let  $F'$  be the set of lifted edges, that is

$$F' = \bigcup_{\substack{f \in F, \\ x_{c(f)} \notin f}} \left\{ \text{two edges joining } x_{c(f)} \text{ to the two vertices of } f \right\},$$

where  $\dot{\cup}$  denotes the disjoint union. Let the multigraph  $D'$  be defined on the same vertex set as  $D$ , and let its edge set be

$$E(D') = \{e \in E(D) : e \notin F \text{ or } x_{c(e)} \notin e\} \dot{\cup} F'.$$

In other words,  $D'$  is the demand graph into which  $D$  is transformed by lifting the elements of  $F$ . Let  $Y = V(D) \setminus X$ . Observe that  $d_{D'}(y) = d_D(y)$

for  $y \in Y$ . Let  $Y_i = \{y \in Y \setminus B \mid c(y) = i\}$  be the color  $i$  vertices in  $Y \setminus B$ . The balancedness of  $c$  guarantees that

$$|Y_i| = |c^{-1}(i) \setminus \{x_{i-1}\} \setminus B| \geq |c^{-1}(i)| - 1 - |B| \geq \left\lfloor \frac{n}{3} \right\rfloor - 5.$$

In the second phase, our task is to resolve all multiplicities of  $x_i$  in  $D'$ . Observe that as edges of  $F$  of the same color formed a matching, out of every two parallel edges that are incident to  $x_i$  in  $D'$  at least one of them must be an initial edge in  $E(D') \setminus F'$ . The vertex  $x_i$  is incident to  $d_{D'}(x_i) - d_{F'}(x_i)$  edges of  $E(D') \setminus F'$ ; we plan to lift these edges to the elements of  $Y_i$  by using every vertex in  $Y_i$  for lifting at most once. Since

$$d_{D'}(x_i) - d_{F'}(x_i) \leq d_D(x_i) - 1 = \Delta(D) - 1 \leq \left\lfloor \frac{n}{3} \right\rfloor - 5 \leq |Y_i|,$$

and elements of  $Y_i$  are not incident to edges of color  $i$ , the set  $Y_i$  offers enough space to carry out the liftings. That being said, note that neighbors of  $x_i$  in  $Y_i$  cannot be used for lifting as they would create additional multiplicities. On the other hand, if  $v \in Y_i$  and  $e = vx_i \in E(D)$  then  $e$  is an initial edge of  $x_i$  that either generates no multiplicity at all or it is part of a bundle of parallel edges, one of which we do not lift. In other words, for every vertex of  $Y_i$  that is excluded from the lifting we mark an initial edge of  $x_i$  that we do not need to lift. As a result of this, resolution of the remaining multiplicities at  $x_i$  can be performed in  $Y_i - \Gamma(x_i)$ . Let  $D''$  denote the demand graph obtained after resolving all of the multiplicities of  $x_1, x_2$ , and  $x_3$ .

At most 1 element of  $E(D') \setminus E(F')$  has been lifted to each  $y \in Y$ , therefore there are no multiple edges between the sets  $X$  and  $Y$  in the demand graph  $D''$ . Moreover,  $D''[X] = D'[X]$  is a subgraph of a triangle, which emerges as we lift the at most one edge of color  $i + 2$  of  $x_i$  to  $x_{i+1}$  (take the indices cyclically), for  $i = 1, 2, 3$ .

Any vertex  $y \in Y$  of color  $i$  has at most two incident edges in  $F'$ , joining  $y$  to a subset of  $\{x_{i+1}, x_{i+2}\}$ .

- If an edge has been lifted to  $y \in Y$  of color  $i$ , then  $y$  is adjacent to  $x_i$  and  $d_{D''}(y) = d_{D'}(y) + 2$ . Thus  $y$  is joined to at least  $d_{F'}(y) + 1$  elements of  $X$  in  $D''$ . As no edge of color  $i$  can be incident to  $y$ , we have  $d_{F'}(y) = d_F(y)$ . Therefore

$$d_{D''[Y]}(y) \leq d_{D''}(y) - d_{F'}(y) - 1 = d_{D'}(y) - d_F(y) + 1 = d_D(y) - d_F(y) + 1.$$

- If no edges have been lifted to  $y \in Y$ , then  $d_{D''}(y) = d_{D'}(y)$  and  $y$  is adjacent to at least  $d_F(y)$  elements of  $X$  in  $D''$ . Therefore

$$d_{D''[Y]}(y) = d_{D''}(y) - d_F(y) \leq d_{D'}(y) - d_F(y) = d_D(y) - d_F(y).$$

As elements of  $B$  are excluded from  $Y_i$ , 0 edges are lifted to them, and so we proved the statement of the lemma.  $\square$

Let  $X_1 = \{x_1, x_2, x_3\}$  be a subset of 3 elements of  $V(D)$ , such that  $D[X]$  has 0 edges. Such a set trivially exists, as any two non-adjacent vertices have  $(n-2) - 2\Delta(D) \geq \frac{n}{3} + 2$  common non-neighbors. Since the degree in  $D$  is at least  $2 \cdot (24/6) - 4 = 4$ , Theorem 7 implies the existence of two disjoint 2-factors,  $A_1$  and  $A_2$  of  $D$ . Notice that  $A_2 - X$  has 3 path components (as a special case, an isolated vertex is a path on one vertex). Extend  $A_2 - X$  to a maximal  $\leq 2$ -factor  $F_2$  of  $D - A_1$ . It is easy to see that there exists a 3-element subset  $B_1$  of  $V(D) \setminus X$  such that

- $B_1$  induces 0 edges in  $D - A_1$ ,
- $\{v \in V(D) \setminus X : d_{F_2}(v) = 0\} \subset B_1$ , and
- $B_2 = \{v \in V(D) \setminus X : d_{F_2}(v) = 1\} \setminus B_1$  has cardinality at most 3.

We are ready to use Lemma 10. First, apply it to  $D$ , where we lift  $F = A_1$  to elements of  $X = X_1$ , while not creating new edges incident to  $B = B_1$ . Let the obtained graph be  $H_1$ . We have  $\Delta(H_1) \leq \Delta(D) - \delta(A_1) + 1 = \Delta(D) - 1$ . Furthermore,  $E(H_1[B_1]) \subseteq E(D[B_1]) = \emptyset$ .

We apply Lemma 10 once more. Now  $H_1$  is our base demand graph,  $F_2$  is the  $\leq 2$ -factor to be lifted to elements of  $B_1$ , and we avoid lifting to elements of  $B_2$ . Let the resulting demand graph be  $H_2$ , whose vertex set is  $V(D) \setminus X \setminus B_1$  of cardinality  $n - 6$ . We have

$$\begin{aligned} d_{H_2}(v) &\leq \begin{cases} d_{H_1}(v) - d_{F_2}(v) + 1 & \text{if } v \notin B_2, \\ d_{H_1}(v) - d_{F_2}(v) & \text{if } v \in B_2. \end{cases} \leq \\ &\leq \begin{cases} (\Delta(D) - 1) - 2 + 1 & \text{if } v \notin B_2, \\ (\Delta(D) - 1) - 1 & \text{if } v \in B_2. \end{cases} \leq \\ &\leq \Delta(D) - 2 = 2 \left\lfloor \frac{n-6}{6} \right\rfloor - 4. \end{aligned}$$

By induction on  $n$ , we know that  $H_2$  has a resolution, implying that  $H_1$  has a resolution, which in turn implies that  $D$  has a resolution.

### 3. PROOF OF THEOREM 6

We prove our statement by induction on  $n$ . For  $n \leq 4$  the statement is straightforward, the cases  $n = 5, 6$  require a somewhat cumbersome case-work. Note that if  $n \geq 4$  we may assume  $D$  has exactly  $2n - 5$  edges, otherwise we join two non-neighbors whose degree is smaller than  $n - 1$ .

For the inductive step, we choose a vertex  $x$ , resolve all of its multiplicities, and delete it from the demand graph. There are two additional conditions to assert as the number of vertices decreases from  $n$  to  $n - 1$ :

- i) We need to delete at least 2 edges from  $D$ . These edges can be either already incident to  $x$  or can be lifted to  $x$ .
- ii) Let  $B$  denote the set of vertices of degree greater than or equal to  $n - 1$ . Obviously, to apply induction we need to decrease the degree  $d(v)$  of every vertex  $v \in B$  by at least one. Decreasing  $d(v)$



can be performed by lifting an edge incident to  $v$  to  $x$ . Note that this operation might create additional multiplicities that need to be resolved before the deletion of  $x$ .

In addition, observe that we can lift at least one edge to a vertex  $v$  without its degree exceeding the degree bound for  $n' = n - 1$  if and only if  $d(v) < n - 2$ . Let

$$B = \{z_1, \dots, z_{|B|}\} = \{v \in V(D) : d(v) \geq n - 2\}.$$

As  $\sum_{v \in V(D)} d(v) = 4n - 10$ , it follows that  $|B| \leq 3$ . We perform a casework on  $|B|$ .

$|B| = 0$ : If  $B$  is empty, then the only condition we need to guarantee is the deletion of at least two edges in  $D$ . We have two cases.

- *If there is an  $x \in V(D)$  with  $\gamma(x) \geq 2$* : we have  $n - 1 - \gamma(x)$  vertices for lifting to resolve the  $d(x) - \gamma(x)$  multiplicities of  $x$ . Obviously,  $d(x) - \gamma(x) \leq n - 3 - \gamma(x)$  thus we have enough space to resolve all multiplicities of  $x$ . After the deletion of  $x$ , the graph has lost  $\gamma(x) \geq 2$  edges, and the maximum degree is still two less than the number of vertices.
- *If  $\forall x \in V(D)$  we have  $\gamma(x) \leq 1$* , then  $D$  is the disjoint union of bundles and isolated vertices, which is trivial to resolve.

$|B| = 1$ : We perform the same operation as in the previous case with the choice  $x = z_1$ . Observe that our inequality becomes  $d(z_1) - \gamma(z_1) \leq n - 1 - \gamma(z_1)$  thus we have enough vertices in the multigraph to perform all the necessary liftings.

$|B| = 2$ : Observe first that  $z_1$  and  $z_2$  are joined by an edge  $e$  or else

$$2n - 5 \geq d(B, V(D) - B) = d(z_1) + d(z_2) \geq 2n - 4,$$

a contradiction. Let us first assume that  $z_1$  or  $z_2$  (say,  $z_1$ ) has an edge ending in a vertex different from  $z_2$  (i.e.  $d(B, V(D) - B) > 0$ ). Observe that in this case  $m(z_1) = d(z_1) - \gamma(z_1) \leq (n - 1) - \gamma(z_1)$ , thus all multiplicities of  $z_1$  can be resolved by lifting the appropriate edges to  $V(D) - \{z_1\} - \Gamma(z_1)$ .

In the remaining case  $z_1$  and  $z_2$  form a bundle of at most  $n - 1$  edges. We can lift  $n - 2$  of these edges to  $V(D) - B$  without difficulties, delete one of the vertices in  $B$ , and proceed by induction.

$|B| = 3$ : Observe that any two vertices of  $\{z_1, z_2, z_3\}$  must be joined by an edge else the same reasoning as above leads to contradiction. Note also that a simple average degree calculation guarantees the existence of an isolated vertex  $x$ . We distinguish two cases:

- i) If  $d(B, V(D) - B) = 0$ , we may assume that  $V(D) - B$  contains an edge, otherwise  $3(n - 3) \geq 4n - 10 \Rightarrow n \leq 7$  and all edges are contained in  $B$ . For  $n = 5, 6, 7$  that leads to 4 possible

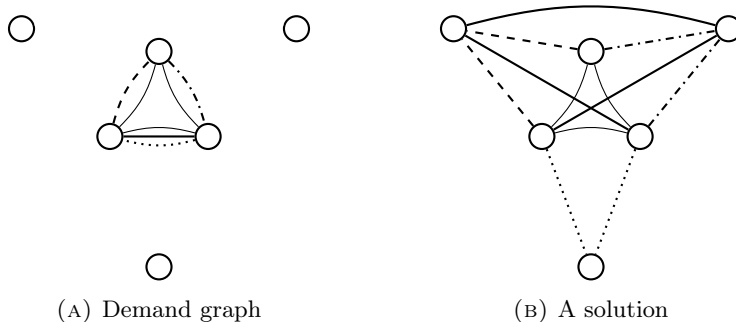


FIGURE 4

demand graphs whose resolution can be easily completed; a case for  $n = 6$  is shown in Figure 4.

Let  $f$  denote an arbitrary edge in  $V(D) - B$ . We lift two edges of  $B$  not belonging to the same pair as well as  $f$  to  $x$ ; observe that the degrees of all vertices in  $B$  dropped by at least 1. As  $n \geq 7$ , the multiple edge created at vertex  $x$  can be lifted to a vertex of  $V(D) - B$  that was not incident to  $f$ .

- ii) If  $d(B, V(D) - B) > 0$  let  $f$  be an edge between  $B$  and  $V(D) - B$ . Without loss of generality we may assume  $f$  is incident to  $z_3$ . We lift  $f$  as well as an edge  $e$  between  $z_1$  and  $z_2$ ; as  $e$  and  $f$  are disjoint, no new multiplicity is created, thus we can proceed by induction.

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